

Uniform Probability and Natural Density of Mutually Left Coprime Polynomial Matrices over Finite Fields

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Abstract

We use the uniform probability distribution as well as the natural density to calculate the probability that finitely many polynomials are pairwise coprime. It will turn out that the formulas for the two considered probability measures asymptotically coincide but differ in the exact values. Moreover, we compute the natural density of mutually left coprime polynomial matrices and compare the result with the formula one gets using the uniform probability distribution. The achieved estimations are not as precise as in the scalar case but again we can show asymptotic coincidence.

1 Introduction

Polynomial matrices over finite fields play an important role in various mathematical areas, e.g. for the investigation of discrete-time linear systems [10], [1] or in the theory of convolutional codes [11]. For many of these applications, coprimeness conditions for the considered matrices are essential, [1].

A polynomial matrix $D \in \mathbb{F}[z]^{n \times m}$ is called left prime if there exists $X \in \mathbb{F}[z]^{m \times n}$ with $DX = I$, where I denotes the identity matrix. It is easily shown that this is equivalent to the condition that the fullsize minors of D are coprime; see e.g. [14]. In this paper, we will need another characterization of left primeness, namely that D has to be of full row rank for every $z \in \overline{\mathbb{F}}$, which clearly is equivalent to the fact that it can be completed to

a unimodular matrix, i.e. to a matrix with nonzero constant determinant. That it is possible to characterize left primeness by this last condition is part of the famous Quillen-Suslin theorem, also known as Serre conjecture [5], which was formulated in 1957 for polynomial matrices in several variables z_1, \dots, z_k . Already in 1958, Seshadri [12] proved its correctness in principal ideal domains and therefore, in the cases $k = 1$ and $k = 2$. The final proof for the general case followed in 1976 [9], [13].

We use the one-dimensional version of this theorem to compute the probability of left primeness for specially structured polynomial matrices using two different probability measures, namely uniform probability and natural density. For the case $n = 1$, i.e. for matrices consisting only of one row, the probability of left primeness coincides with the probability of coprimeness for scalar polynomials, which was computed in [2] to be equal to $1 - t^{m-1}$, where $t := |\mathbb{F}|^{-1}$. For matrices of arbitrary sizes, Guo and Yang [3] computed the natural density of left primeness to be equal to $\prod_{j=m-n}^{m-1} (1 - t^j)$, using techniques from [8], where this computation was done for integer matrices.

In Theorem 9 of [6], the probability that a matrix of the form $[D_1 \ D_2] \in \mathbb{F}[z]^{m \times 2m}$ with $\deg(\det(D_i)) = n_i \in \mathbb{N}$ is left prime, i.e. that $D_1 \in \mathbb{F}[z]^{m \times m}$ and $D_2 \in \mathbb{F}[z]^{m \times m}$ are left coprime, was calculated. It turns out that the obtained formula, namely $1 - t^m + O(t^{m+1})$ for $t \rightarrow 0$, asymptotically coincides with the formula for the natural density of left primeness for an arbitrary polynomial matrix from $\mathbb{F}[z]^{m \times 2m}$, computed in [3] to be equal to $\prod_{j=m}^{2m-1} (1 - t^j)$.

According to Proposition 10.3 of [1], the property of N matrices from $\mathbb{F}[z]^{m \times m}$ to be mutually left coprime is equivalent to the left primeness of a specially structured matrix from $\mathbb{F}^{(N-1)m \times mN}$. In [4], the uniform probability of mutual left coprimeness was calculated for scalar polynomials with fixed degrees, i.e. for $m = 1$, where mutual left coprimeness and pairwise coprimeness coincide. This result was generalized in [6], obtaining a probability of $1 - \sum_{y=2}^{m+1} \binom{N}{y} t^y + O(t^{m+1})$ for the probability of mutual left coprimeness for N matrices from $\mathbb{F}[z]^{m \times m}$ whose degrees of the determinant are fixed. In this article, we firstly improve the estimation for the case $m = 1$ and secondly, compute the natural density of mutual left coprimeness in the cases $m = 1$ and $m \in \mathbb{N}$. It will turn out that the formulas for uniform probability distribution and natural density asymptotically coincide in all computed cases. The paper is structured as follows. In Section 2, we provide some basic definitions, properties and formulas, which we will need in the following sections.

Section 3 deals with the case $m = 1$, i.e. uniform probability and natural density of pairwise coprime polynomials are calculated. It turns out that the obtained asymptotic expressions for uniform probability and natural density coincide. After that, in Section 4, we prove our main result, Theorem 4.8, which provides an asymptotic formula for the natural density of mutually left coprime polynomial matrices. Finally, we compare this result with the uniform probability that polynomial matrices are mutually left coprime and could again observe asymptotical identicalness.

2 Preliminaries

2.1 Coprimeness of Polynomial Matrices

In this subsection, we will provide some basic definitions and properties concerning polynomial matrices over an arbitrary field \mathbb{F} .

Definition 2.1.

A polynomial matrix $Q \in \mathbb{F}[z]^{m \times m}$ is called **nonsingular** if $\det(Q(z)) \neq 0$. It is called **unimodular** if $\det(Q(z)) \neq 0$ for all $z \in \overline{\mathbb{F}}$, i.e. if $\det(Q(z))$ is a nonzero constant. This is the case if and only if Q is invertible in $\mathbb{F}[z]^{m \times m}$. Hence, one denotes the group of unimodular $m \times m$ -matrices over $\mathbb{F}[z]$ by $Gl_m(\mathbb{F}[z])$.

Definition 2.2.

A polynomial matrix $H \in \mathbb{F}[z]^{p \times m}$ is called a **common left divisor** of $H_i \in \mathbb{F}[z]^{p \times m_i}$ for $i = 1, \dots, N$ if there exist matrices $X_i \in \mathbb{F}[z]^{m \times m_i}$ with $H_i(z) = H(z)X_i(z)$ for $i = 1, \dots, N$. It is called a **greatest common left divisor**, which is denoted by $H = \text{gcd}(H_1, \dots, H_N)$, if for any other common left divisor $\tilde{H} \in \mathbb{F}[z]^{p \times \tilde{m}}$ there exists $S(z) \in \mathbb{F}[z]^{\tilde{m} \times m}$ with $H(z) = \tilde{H}(z)S(z)$. A polynomial matrix $E \in \mathbb{F}[z]^{p \times m}$ is called a **common left multiple** of $E_i \in \mathbb{F}[z]^{m_i \times m}$ for $i = 1, \dots, N$ if there exist matrices $X_i \in \mathbb{F}[z]^{p \times m_i}$ with $X_i(z)E_i(z) = E(z)$ for $i = 1, \dots, N$. It is called a **least common left multiple**, which is denoted by $E = \text{lcm}(E_1, \dots, E_N)$, if for any other common left multiple $\tilde{E} \in \mathbb{F}[z]^{\tilde{p} \times m}$, there exists $R(z) \in \mathbb{F}[z]^{\tilde{p} \times p}$ with $R(z)E(z) = \tilde{E}(z)$. One defines a **(greatest) common right divisor**, which is denoted by gcdr , and a **(least) common right multiple**, which is denoted by lcrm , analogously.

Definition 2.3.

Polynomial matrices $H_i \in \mathbb{F}[z]^{p \times m_i}$ are called **left coprime** if there exists $X \in \mathbb{F}[z]^{m \times p}$ such that $H = \text{gcd}(H_1, \dots, H_N)$ satisfies $HX = I_p$. In particular, one polynomial matrix $H \in \mathbb{F}[z]^{p \times m}$ is called **left prime** if there exists $X \in \mathbb{F}[z]^{m \times p}$ with $HX = I_p$. Analogously, one defines the property to be **right coprime** or **right prime**, respectively. Note that in the case $p = m$, right primeness and left primeness are equivalent to the property to be unimodular.

Theorem 2.4.

The polynomial matrices $H_i \in \mathbb{F}[z]^{p \times m_i}$ are left coprime if and only if $\text{rk}(H_1(z), \dots, H_N(z)) = p$ for all $z \in \overline{\mathbb{F}}[z]$.

Remark 2.5.

- (a) Left coprimeness of $H_i \in \mathbb{F}[z]^{p \times m_i}$ is equivalent to left primeness of the matrix (H_1, \dots, H_N) .
- (b) A rectangular matrix $H \in \mathbb{F}[z]^{p \times m}$ with $p \leq m$ is left prime if and only if its $p \times p$ -minors are coprime.

Definition 2.6.

Nonsingular polynomial matrices $D_1, \dots, D_N \in \mathbb{F}[z]^{m \times m}$ are called **mutually left coprime** if for each $i = 1, \dots, N$, D_i is left coprime with $\text{lcrm}\{D_j\}_{j \neq i}$.

Theorem 2.7. [1, Proposition 10.3]

Nonsingular polynomial matrices $D_1, \dots, D_N \in \mathbb{F}[z]^{m \times m}$ are mutually left coprime if and only if

$$\mathcal{D}_N := \begin{bmatrix} D_1 & D_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & D_{N-1} & D_N \end{bmatrix}$$

is left prime.

2.2 Probability Distributions and Basic Counting Formulas

To compute the probability that a mathematical object has a special property, it is necessary to count mathematical objects. Therefore, in the following, we restrict our considerations to a finite field \mathbb{F} with cardinality $|\mathbb{F}|$.

Firstly, \mathbb{F} should be endowed with the uniform probability distribution that assigns to each field element the same probability

$$t = \frac{1}{|\mathbb{F}|}.$$

In addition to computing probabilities with the uniform distribution, which is only defined for finite sets, we will compare these results with the results one gets using another definition of probability, namely the concept of natural density as defined in [3] for infinite sets:

Definition 2.8.

To enumerate $\mathbb{F}[z]$, assign the number $k = \sum_{i=0}^{\infty} a_i (\frac{1}{t})^i$ to the polynomial $f_k(z) = \sum_{i=0}^{\infty} a_i z^i$. In particular, $f_0 \equiv 0$. Moreover, let \mathcal{M}_n be the set of tuples $(D_1, \dots, D_N) \in (\mathbb{F}[z]^{l \times m})^N$ for which the entries of D_i are elements of the set $\{f_0, \dots, f_n\}$ for $i = 1, \dots, N$. The natural density of a set $E \subset (\mathbb{F}[z]^{l \times m})^N$ in $(\mathbb{F}[z]^{l \times m})^N$ is defined as $\lim_{n \rightarrow \infty} \frac{|E \cap \mathcal{M}_n|}{|\mathcal{M}_n|}$.

Moreover, for later computations, we will need the following lemmata, which provide well-know formulas for the determination of cardinalities.

Lemma 2.9. [7, S. 455]

For $1 \leq r \leq \min(k, n)$, denote by $N(k, n, r)$ the number of matrices from $\mathbb{F}^{k \times n}$ that have rank r . Then, it holds

$$N(k, n, r) = t^{-nr} \prod_{i=n-r+1}^n (1 - t^i) \cdot \prod_{i=0}^{r-1} \frac{t^{i-k} - 1}{t^{-(i+1)} - 1}.$$

In particular, the number of invertible $n \times n$ -matrices over \mathbb{F} is equal to

$$|Gl_n(\mathbb{F})| = t^{-n^2} \prod_{j=1}^n (1 - t^j).$$

Lemma 2.10. (Inclusion-Exclusion Principle)

Let A_1, \dots, A_n be finite sets and $X = \bigcup_{i=1}^n A_i$. For $I \subset \{1, \dots, n\}$, define $A_I := \bigcap_{i \in I} A_i$. Then, it holds

$$|X| = \sum_{\emptyset \neq I \subset \{1, \dots, n\}} (-1)^{|I|-1} |A_I|.$$

Lemma 2.11. [2]

The probability that N monic polynomials $d_1, \dots, d_N \in \mathbb{F}[z]$ with $\deg(d_i) = n_i \in \mathbb{N}$ for $i = 1, \dots, N$ are coprime is equal to $1 - t^{N-1}$.

Lemma 2.12.

The number of monic irreducible polynomials in $\mathbb{F}[z]$ of degree j is equal to

$$\varphi_j = \frac{1}{j} \sum_{d|j} \mu(d) t^{-j/d} = \frac{1}{j} t^{-j} + O(t^{-(j-1)}),$$

where μ counts the number of distinct prime factors of an integer and is zero if the integer is the multiple of a square-number.

Lemma 2.13.

For $M \in \mathbb{N}$, it holds

$$\sum_{k=1}^M (-1)^k \frac{1}{k!(M-k)!} = -\frac{1}{M!}.$$

Proof. Using the binomial formula, one obtains

$$0 = (1-1)^M = \sum_{k=0}^M \binom{M}{k} (-1)^k \Leftrightarrow 0 = \sum_{k=0}^M (-1)^k \frac{1}{k!(M-k)!}.$$

□

3 Counting Pairwise Coprime Polynomials

According to [4, Corollary 3], the probability that N monic polynomials $d_i \in \mathbb{F}[z]$ with $\deg(d_i) = n_i \in \mathbb{N}$ for $i = 1, \dots, N$ are pairwise coprime is

$$1 - \frac{N(N-1)}{2} \cdot t + O(t^2) \tag{1}$$

if $1/t$ tends to infinity. The method used in [4] to prove this result has the advantage that in principle, it is possible to compute the coefficients of t^j for $j \geq 2$ in this asymptotic expansion with the same procedure. But when j is increasing, the computational effort for doing this becomes very large. In the following, we want to improve this estimation by additionally computing

the coefficient of t^2 . Prior to this, we need to introduce some notation, which was also used in [4].

First, a more general setup should be considered. Let $n := (n_1, \dots, n_N) \in \mathbb{N}^N$ and Γ be an undirected graph with set of vertices $\mathcal{V} = \{1, \dots, N\}$ and set of edges \mathcal{E} , having cardinality $E := |\mathcal{E}|$. The edges of Γ are denoted as ij , for suitable $i, j \in \mathcal{V}$ with $i < j$. For every vertex $l \in \mathcal{V}$ let

$$\mathcal{E}_l := \{ij \in \mathcal{E} \mid i = l \text{ or } j = l\}$$

denote the set of edges terminating at l . Moreover, gcd and lcm should denote the monic greatest common divisor and least common multiple, respectively. Let $X(n) := \{(d_1, \dots, d_N) \mid d_i \in \mathbb{F}[z] \text{ monic with } \deg(d_i) = n_i\}$ and $\Gamma(n) := \{(d_1, \dots, d_N) \in X(n) \mid \gcd(d_i, d_j) = 1 \text{ for } ij \in \mathcal{E}\}$. Clearly, $|X(n)| = t^{-(n_1 + \dots + n_N)}$. With each edge ij of Γ we associate a monic, square-free polynomial $k_{ij}(z) \in \mathbb{F}[z]$. We refer to this as a polynomial labeling of the graph and denote it by \mathbf{k} . For each polynomial labeling and vertices $l \in \mathcal{V}$, let

$$K_l := \text{lcm}\{k_{ij} \mid ij \in \mathcal{E}_l\}.$$

Then

$$M(n) := \{\mathbf{k} \in \mathbb{F}[z]^E \mid k_{ij} \text{ monic, square-free for } ij \in \mathcal{E}, \deg(K_l) \leq n_l, l \in \mathcal{V}\}$$

is the set of all polynomial labelings \mathbf{k} of Γ satisfying the degree bounds $\deg(K_l) \leq n_l$ for all vertices l . For each monic square-free polynomial p , let $\omega(p)$ denote the number of irreducible factors of p . To achieve formula (1) as well as our improvement, the following exact expression for the considered probability is used:

Theorem 3.1. [4, Theorem 5]
The cardinality of $\Gamma(n)$ is

$$|\Gamma(n)| = t^{-(n_1 + \dots + n_N)} \sum_{\mathbf{k} \in M(n)} \prod_{ij \in \mathcal{E}} (-1)^{\omega(k_{ij})} \prod_{l=1}^N t^{\deg(K_l)}. \quad (2)$$

In the case that all pairs of vertices of Γ are connected by an edge, one obtains the probability that N monic polynomials are pairwise coprime. Next, the preceding theorem is used to improve the estimation from formula (1):

Theorem 3.2.

Let $n_1, \dots, n_N \in \mathbb{N}$ and $N_1 := |\{l \in \{1, \dots, N\} \mid n_l = 1\}|$. Then, the probability that N monic polynomials over \mathbb{F} of degrees n_1, \dots, n_N are pairwise coprime is equal to

$$1 - \frac{N(N-1)}{2} \cdot t + \frac{1}{24}(N-1)(N-2)(3N^2 + 11N - 12N_1) \cdot t^2 + O(t^3).$$

Proof. Let G be a graph with N vertices, which are pairwise connected by an edge and let $|G(n)|$ be the number of N -tuples of monic pairwise coprime polynomials over \mathbb{F} of degrees n_1, \dots, n_N . Moreover, Γ should be any subgraph of G , whose number of edges is equal to E . To prove the result, we first consider the general graph Γ and sort the elements of $M(n)$ with respect to the degrees of the entries of the vector $\mathbf{k} = (k_1, \dots, k_E)$.

To this end, for each vector of non-negative integers $\mathbf{g} := (g_1, \dots, g_E)$ define $M(n, \mathbf{g}) := \{\mathbf{k} \in M(n) \mid \deg(k_m) = g_m \text{ for } 1 \leq m \leq E\}$. Let A be the set of all \mathbf{g} with $M(n, \mathbf{g}) \neq \emptyset$. Note that the degree bounds for $M(n)$ ensure that A is finite. One achieves:

$$|\Gamma(n)| = t^{-(n_1 + \dots + n_N)} \sum_{\mathbf{g} \in A} \sum_{\mathbf{k} \in M(n, \mathbf{g})} \prod_{ij \in \mathcal{E}} (-1)^{\omega(k_{ij})} \prod_{l=1}^N t^{\deg(K_l)}.$$

Now consider G , i.e. the case $E = \frac{N(N-1)}{2}$. Starting with small values for the entries of \mathbf{g} , the first summands are computed.

For $\mathbf{g} = (0, \dots, 0)$, i.e. $\mathbf{k} = (1, \dots, 1)$, one gets the summand 1 because of $\omega(1) = 0$ and $K_l = 1$ for $l = 1, \dots, N$. If $g_{m_0} = 1$ for exactly one $1 \leq m_0 \leq E$ and $g_m = 0$ for $m \neq m_0$, there are $|\mathbb{F}| = 1/t$ possibilities for the linear polynomial k_{m_0} and E possibilities for the choice of m_0 . Moreover, $\omega(k_{m_0}) = 1$, so that these summands have negative sign. As k_{m_0} is relevant for exactly those K_l for which its associated edge is terminating at l , there are exactly two K_l which are of degree 1. Hence, the resulting sum of these terms is equal to $-E \cdot \frac{1}{t} \cdot t^2 = -E \cdot t$.

Note that for all summands computed so far, every \mathbf{k} lies in $M(n, \mathbf{g})$ since $\deg(K_l) \leq 1$ in all considered cases. Next look at the summands whose sum of the entries of \mathbf{g} is equal to 2. All summands for which $g_{m_0} = 2$ for exactly one $1 \leq m_0 \leq E$ and $g_m = 0$ for $m \neq m_0$ have modulus t^4 . They have negative sign if k_{m_0} is irreducible and positive sign otherwise, it follows from Remark 1 of [4] that these summands add up to zero. Hence, in this case, it does not matter whether \mathbf{k} lies in $M(n, \mathbf{g})$ or not since this depends only on

m_0 and not on k_{m_0} itself.

Now consider the summands for which two entries of \mathbf{g} are equal to one, and the other entries are equal to zero. If the corresponding edges of the nonzero entries have a vertex l in common, the summand has the value

$$\begin{aligned} \sum_{\substack{k_1, k_2 \text{ monic} \\ \deg(k_m)=1}} t^{2+\deg(\text{lcm}(k_1, k_2))} &= \sum_{\substack{k_1 = k_2 \text{ monic} \\ \deg(k_m)=1}} t^3 + \sum_{\substack{k_1 \neq k_2 \text{ monic} \\ \deg(k_m)=1}} t^4 = \\ &= \frac{1}{t} \cdot t^3 + \frac{1}{t} \cdot \left(\frac{1}{t} - 1 \right) \cdot t^4 = 2t^2 - t^3 \end{aligned} \quad (3)$$

if $n_l \geq 2$ and t^2 if $n_l = 1$ since the summands of the second sum lie not in $M(n, \mathbf{g})$ if $\deg(K_l) = 2 > n_l$. For such an "angle", there are $N \cdot \binom{N-1}{2}$ possibilities, N for the apex and $\binom{N-1}{2}$ for the two sides of the angle. If those two edges are isolated, the summand has the value

$$\sum_{\substack{k_1, k_2 \text{ monic} \\ \deg(k_m)=1}} t^4 = t^2.$$

For this case, there are $\binom{N}{4}$ possibilities to choose the 4 involved vertices and 3 possibilities to connect two of them, pairwise.

In summary, all summands whose sum of the entries of \mathbf{g} is equal to two contribute the value

$$\begin{aligned} &\left(\binom{N-1}{2} (2(N - N_1) + N_1) + 3 \cdot \binom{N}{4} \right) \cdot t^2 + O(t^3) = \\ &\left(\frac{(N-1)(N-2)}{2} (2N - N_1) + \frac{N(N-1)(N-2)(N-3)}{8} \right) \cdot t^2 + O(t^3) = \\ &\frac{(N-1)(N-2)}{8} (N^2 + 5N - 4N_1) \cdot t^2 + O(t^3). \end{aligned} \quad (4)$$

If three entries of \mathbf{g} are equal to 1, where the corresponding edges form a triangle, and the other entries are equal to 0, one gets

$$-\frac{1}{t} \cdot t^3 - \frac{3}{t} \cdot \left(\frac{1}{t} - 1 \right) \cdot t^5 - \frac{1}{t} \cdot \left(\frac{1}{t} - 1 \right) \cdot \left(\frac{1}{t} - 2 \right) \cdot t^6 = -t^2 + O(t^3).$$

Here the first summand of the left side of the equation computes the case that three, the second summand that two and the third summand that none of the

three entries of \mathbf{k} that contain a linear polynomial are identical. Moreover, there are $\binom{N}{3}$ possibilities for such a triangle. Adding these summands to (4), one gets

$$\begin{aligned} & \frac{(N-1)(N-2)}{8}(N^2 + 5N - 4N_1 - 4N/3) \cdot t^2 + O(t^3) = \\ & \frac{(N-1)(N-2)}{24}(3N^2 + 11N - 12N_1) \cdot t^2 + O(t^3). \end{aligned}$$

Now we turn to the general graph Γ again and show that

$$R(\mathbf{g}) := \sum_{\mathbf{k} \in M(n, \mathbf{g})} \prod_{l=1}^N t^{\deg(K_l)} = O(t^3)$$

for every fixed \mathbf{g} for which the sum of the entries of \mathbf{g} is at least three and Γ is no triangle.

This will be done by induction with respect to E .

For $E = 1$, note that \mathbf{g} and $\mathbf{k} = k_{12}$ are scalar. Moreover, $K_1 = K_2 = k_{12}$. Therefore, $R(\mathbf{g}) = 0$ if $\mathbf{g} > \min(n_1, n_2)$ and otherwise

$$R(\mathbf{g}) \leq \sum_{\mathbf{k} \text{ monic, } \deg(\mathbf{k})=\mathbf{g}} t^{2 \deg(\mathbf{k})} = \left(\frac{1}{t}\right)^{\mathbf{g}} \cdot t^{2\mathbf{g}} = t^{\mathbf{g}} = O(t^3) \text{ for } \mathbf{g} \geq 3.$$

This computation starts with an inequality since the condition that \mathbf{k} has to be square-free is dropped. The first equality follows from the fact that there are $(1/t)^{\mathbf{g}}$ monic polynomials of degree \mathbf{g} .

Next, we take the step from $E - 1$ to E .

To this end, choose one of the smallest entries of \mathbf{g} and denote it without loss of generality by g_E . Then, the edge with which k_E is associated – in the following denoted by ij – is taken away from the original graph and thus a graph with $E - 1$ edges is achieved. In the following, the index $(E - 1)$ above an expression means that it belongs to a graph with $E - 1$ edges; in the same way we use the index (E) . Similarly, $\mathbf{k}^{(E-1)}$ and $\mathbf{g}^{(E-1)}$ should denote the vectors consisting of the first $E - 1$ entries of \mathbf{k} and \mathbf{g} , respectively.

The degrees of the K_l can never increase, when taking an edge away. Therefore, $\mathbf{k} \in M(n, \mathbf{g})$ implies $\mathbf{k}^{(E-1)} \in M^{(E-1)}(n, \mathbf{g}^{(E-1)})$. Next we set

$$W_i := \gcd(K_i^{(E-1)}, k_E) \quad \text{and} \quad W_j := \gcd(K_j^{(E-1)}, k_E).$$

Moreover, let

$$B_{v_i, v_j}^{(E-1)} := \{\mathbf{k}^{(E-1)} \in M^{(E-1)}(n, \mathbf{g}^{(E-1)}) \mid \deg(K_i^{(E-1)}) = v_i, \deg(K_j^{(E-1)}) = v_j\},$$

$$B_{v_i, v_j, w_i, w_j}^{(E)} := \{\mathbf{k} \in M^{(E)}(n, \mathbf{g}) \mid \mathbf{k}^{(E-1)} \in B_{v_i, v_j}^{(E-1)}, \deg(W_i) = w_i, \deg(W_j) = w_j\}.$$

It follows

$$R(\mathbf{g}) \leq \sum_{v_i, v_j, w_i, w_j \leq \max(n_i, n_j)} \sum_{\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E)}} \prod_{l=1}^N t^{\deg(K_l^{(E)})}.$$

The number of summands in the first sum is finite and thus one only has to show that for any fixed v_i, v_j, w_i, w_j the following is true:

$$\sum_{\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E)}} \prod_{l=1}^N t^{\deg(K_l^{(E)})} = O(t^3).$$

To do this one computes

$$K_i^{(E)} = \text{lcm}(K_i^{(E-1)}, k_E) = \frac{K_i^{(E-1)} \cdot k_E}{W_i}.$$

Consequently, one has $\deg(K_i^{(E)}) = \deg(K_i^{(E-1)}) + g_E - w_i$ and $\deg(K_j^{(E)}) = \deg(K_j^{(E-1)}) + g_E - w_j$, analogously. For $l \notin \{i, j\}$ it holds $K_l^{(E)} = K_l^{(E-1)}$ because nothing changes at the associated vertices. It follows:

$$\sum_{\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E)}} \prod_{l=1}^N t^{\deg(K_l^{(E)})} = \sum_{\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E-1)}} \prod_{l=1}^N t^{\deg(K_l^{(E-1)})} \cdot t^{2g_E - w_i - w_j}. \quad (5)$$

Here, the product $\prod_{l=1}^N t^{\deg(K_l^{(E-1)})}$ is independent of k_E and $t^{2g_E - w_i - w_j}$ is independent of \mathbf{k} .

Next, for $\mathbf{k}^{(E-1)} \in B_{v_i, v_j}^{(E-1)}$, an upper bound for the number of polynomials k_E such that $\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E)}$ should be determined. $\mathbf{k}^{(E-1)}$ uniquely determines $K_i^{(E-1)}$ and since W_i is a divisor of $K_i^{(E-1)}$ of degree w_i , there are only finitely many possibilities for W_i . Define C as this number of possibilities for W_i . One knows that k_E has to be a multiple of W_i of degree g_E . Thus, for each W_i there are at most $t^{w_i - g_E}$ possibilities for k_E .

Using this and the fact that the product in (5) is independent of k_E , it follows for the expression in (5):

$$\begin{aligned}
\sum_{\mathbf{k} \in B_{v_i, v_j, w_i, w_j}^{(E)}} \prod_{l=1}^N t^{\deg(K_l^{(E)})} &\leq t^{2g_E - w_i - w_j} \cdot C t^{w_i - g_E} \sum_{\mathbf{k}^{(E-1)} \in B_{v_i, v_j}^{(E-1)}} \prod_{l=1}^N t^{\deg(K_l^{(E-1)})} \\
&= C t^{g_E - w_j} \sum_{\mathbf{k}^{(E-1)} \in B_{v_i, v_j}^{(E-1)}} \prod_{l=1}^N t^{\deg(K_l^{(E-1)})} \\
&\leq C \cdot R(\mathbf{g}^{(E-1)})
\end{aligned}$$

because $w_j \leq g_E$ since $W_j \mid k_E$. Now we distinguish several cases:

Case 1: The sum of the entries of $\mathbf{g}^{(E-1)}$ is at least three and no triangle. Then, $R(\mathbf{g}^{(E-1)})$ is $O(t^3)$ per induction and we are done.

Case 2: $\mathbf{g}^{(E-1)} = (1, 1, 1)$ and $\Gamma^{(E-1)}$ is a triangle.

This case can be avoided: It holds $\mathbf{g}^{(\mathbf{E})} = (1, 1, 1, 1)$ since $\mathbf{g}^{(\mathbf{E})} = (1, 1, 1, 0)$ would mean that $\Gamma^{(E)}$ is a triangle, too, because an edge ij with labelling $k_{ij} = 1$ could be treated like it would not exist. Therefore, one of the vertices of the triangle has an third edge which connects it with the additional vertex. Since all entries of $\mathbf{g}^{(\mathbf{E})}$ are identical, one can take away an arbitrary edge in our process of induction. If one takes away one of the edges which form the triangle, the resulting $\Gamma^{(E-1)}$ is not a triangle any more.

It remains to consider all possible cases for which the sum of the entries of $\mathbf{g}^{(E-1)}$ is smaller than three but the sum of the entries of $\mathbf{g}^{(E)}$ is at least three and $\Gamma^{(E)}$ is no triangle. First, one excludes zero entries in these vectors (case 3) and then, considers $\mathbf{g}^{(E-1)} = (1, 1)$ (case 4) and $\mathbf{g}^{(E-1)} = 2$ (cases 5 and 6).

Case 3: $\mathbf{g}^{(E-1)}$ has a component that is equal to zero.

Here, g_E must be zero since it was choosen to be one of the smallest entries. Thus, $\Gamma^{(E-1)}$ and $\Gamma^{(E)}$ could be treated as being identical and hence, $\Gamma^{(E-1)}$ fulfills the conditions of case 1. Consequently, we are done, too.

Case 4: $\mathbf{g}^{(E)} = (1, 1, 1)$ and $\Gamma^{(E)}$ is no triangle.

Case 4a: $\Gamma^{(E)}$ consists of three isolated edges:

$$R(\mathbf{g}) \leq \left(\frac{1}{t}\right)^3 \cdot t^6 = t^3 = O(t^3).$$

Case 4b: $\Gamma^{(E)}$ consists of an isolated edge and an angle (see (3)):

$$R(\mathbf{g}) \leq \frac{1}{t} \cdot t^2 \cdot (2t^2 - t^3) = O(t^3).$$

Case 4c: $\Gamma^{(E)}$ consists of three edges forming one line:

$$R(\mathbf{g}) \leq \frac{1}{t} \cdot t^4 + \frac{2}{t} \left(\frac{1}{t} - 1 \right) \cdot t^5 + \frac{1}{t} \left(\frac{1}{t} - 1 \right)^2 \cdot t^6 = O(t^3).$$

The first summand covers the case that all linear polynomials in \mathbf{k} are identical, the second summand the case that the polynomial of the edge in the middle coincides with one of the others and the third polynomial is different and the third summand the case that the polynomial in the middle is different from the other two polynomials.

Case 4d: $\Gamma^{(E)}$ consists of three edges that meet at one vertex:

$$R(\mathbf{g}) \leq \frac{1}{t} \cdot t^4 + \frac{3}{t} \left(\frac{1}{t} - 1 \right) \cdot t^5 + \frac{1}{t} \left(\frac{1}{t} - 1 \right) \left(\frac{1}{t} - 2 \right) \cdot t^6 = O(t^3).$$

The first summand covers the case that all linear polynomials in \mathbf{k} are identical, the second summand the case that exactly two of them are identical and the third summand the case that all polynomials are different.

Case 5: $\mathbf{g}^{(E)} = (2, 1)$.

Since we are considering upper bounds for $R(\mathbf{g})$ in the following, we can drop the condition that the quadratic polynomials have to be square-free.

Case 5a: $\Gamma^{(E)}$ consists of two isolated edges:

$$R(\mathbf{g}) \leq \left(\frac{1}{t} \right)^3 \cdot t^6 = O(t^3).$$

Case 5b: $\Gamma^{(E)}$ consists of an angle:

$$R(\mathbf{g}) \leq \frac{1}{t} \left(\frac{1}{t} - 1 \right) \cdot t^5 + \frac{1}{t^3} \cdot t^6 = O(t^3).$$

The first summand covers the case that the linear polynomial divides the quadratic polynomial, the second summand the other case.

Case 6: $\mathbf{g}^{(E)} = (2, 2)$.

Case 6a: G consists of two isolated edges:

$$R(\mathbf{g}) \leq \left(\frac{1}{t}\right)^4 \cdot t^8 = O(t^3).$$

Case 6b: G consists of an angle:

$$R(\mathbf{g}) \leq \frac{1}{t^2} \cdot t^6 + \frac{1}{t^4} \cdot t^7 = O(t^3).$$

The first summand covers the case that the two quadratic polynomials are identical, the second summand the other case.

It follows that $R(\mathbf{g}) = O(t^3)$ for every fixed \mathbf{g} for which the sum of the entries of \mathbf{g} is at least three and Γ is no triangle. Consequently,

$$|G(n)| = t^{n_1 + \dots + n_N} \cdot \left(1 - \frac{N(N-1)}{2} \cdot t + \frac{1}{24}(N-1)(N-2)(3N^2 + 11N - 12N_1) \cdot t^2 + O(t^3)\right).$$

□

So far, we used the uniform probability distribution and fixed the degrees of the considered polynomials. In the following, this result should be compared with the natural density of pairwise coprime polynomials. To this end, we follow the lines of the proof for Theorem 1 in [3].

Theorem 3.3.

The natural density of N polynomials $d_1, \dots, d_N \in \mathbb{F}[z]$ to be pairwise coprime is equal to

$$\prod_{j=1}^{\infty} ((1 - t^j)^{N-1} (1 + (N-1)t^j))^{\varphi_j}.$$

Proof.

From Theorem 2.7, one knows that d_1, \dots, d_N are pairwise coprime if and only if the matrix

$$\mathcal{D}_N := \begin{bmatrix} d_1 & d_2 & 0 & \dots & 0 \\ 0 & d_2 & d_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_{N-1} & d_N \end{bmatrix}$$

is left prime. According to Remark 2.5 (b), this holds if and only if the size $N - 1$ minors of \mathcal{D}_N are coprime.

In the following, the notation of Definition 2.8 is used. Let M_n be the set of all tuples $(d_1, \dots, d_N) \in \mathbb{F}[z]^N$ for which $d_i \in \{f_0, \dots, f_n\}$ for $i = 1, \dots, N$. Furthermore, let \hat{P} be the set of all (monic) irreducible polynomials in $\mathbb{F}[z]$ and P a finite subset of \hat{P} . Moreover, E_P should denote the set of all tuples $(d_1, \dots, d_N) \in \mathbb{F}[z]^N$ for which the gcd of all size $N - 1$ minors of \mathcal{D}_N is coprime with all elements in P . Consequently, we are interested in the probability that $(d_1, \dots, d_N) \in \mathbb{F}[z]^N$ lies in $E := \bigcap_P E_P$; i.e., to obtain the natural density one has to determine $\lim_{n \rightarrow \infty} \frac{|E \cap M_n|}{|M_n|}$.

In a first step, one computes the probability that $(d_1, \dots, d_N) \in M_n$ lies in E_P . To this end, one defines $f_P := \prod_{f \in P} f$ and $d_P := \deg(f_P)$. Next, consider the projection

$$M_n \rightarrow M_n/(f_P) = \prod_{f \in P} M_n/(f)$$

$$(d_1, \dots, d_N) \mapsto (d_1, \dots, d_N)/(f_P) = \prod_{f \in P} (d_1, \dots, d_N)/(f),$$

which applies the canonical projection modulo f_P ($\mathbb{F}[z] \rightarrow \mathbb{F}[z]/(f_P)$) to each entry of (d_1, \dots, d_N) . For $(d_1, \dots, d_N) \in M_n$ holds:

$$\begin{aligned} (d_1, \dots, d_N) &\in E_P \\ \Leftrightarrow \forall f \in P \exists \text{ size } N - 1 \text{ minor of } \mathcal{D}_N \text{ that is not divided by } f \\ \Leftrightarrow \forall f \in P \exists \text{ size } N - 1 \text{ minor of } \mathcal{D}_N \text{ that is nonzero in } (\mathbb{F}[z]/(f))^{N-1 \times N} \\ \Leftrightarrow \forall f \in P : \mathcal{D}_N/(f) \text{ has full rank in } (\mathbb{F}[z]/(f))^{N-1 \times N} \simeq (\mathbb{F}^{\deg f})^{N-1 \times N}, \end{aligned}$$

where $\mathbb{F}^{\deg f}$ denotes the field with $t^{-\deg(f)}$ elements. Note that the matrix

$$\mathcal{D}_N/(f) = \begin{bmatrix} a_1 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{N-1} & a_N \end{bmatrix} \in (\mathbb{F}^{\deg f})^{N-1 \times N}$$

has full rank if and only if $a_i = 0$ for at most one $i \in \{1, \dots, N\}$. The probability for this is equal to $(1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}$.

First, suppose that t^{-d_P} divides $|\{f_0, \dots, f_n\}| = n + 1$, i.e. $n = m t^{-d_P} - 1$ for

some $m \in \mathbb{N}$. Then, one could write $\{f_0, \dots, f_n\} = \{f_s(z)z^{d_P} + f_r(z) \mid 0 \leq s \leq m-1, 0 \leq r \leq t^{-d_P} - 1\}$. One has $\{f_r \mid 0 \leq r \leq t^{-d_P} - 1\} \simeq \mathbb{F}[z]/(f_P)$ and $f_s(z)z^{d_P} + f_r(z) \bmod f_P(z) = f_s(z)z^{d_P} \bmod f_P(z) + f_r(z) = \hat{f}_s(z) + f_r(z)$ where $\hat{f}_s(z) := f_s(z)z^{d_P} \bmod f_P(z) \in \mathbb{F}[z]/(f_P)$. Hence, for every fixed s the canonical projection is bijective and on $\{f_0, \dots, f_n\}$ it is m -to-one. In summary, one obtains

$$\begin{aligned} |E_P \cap M_n| &= m^N \cdot \prod_{f \in P} t^{-N \deg(f)} ((1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}) = \\ &= (m t^{-d_P})^N \cdot \prod_{f \in P} (1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}. \end{aligned}$$

Since $m t^{-d_P} = n + 1$, i.e. $(m t^{-d_P})^N = |M_n|$, it follows

$$\frac{|E_P \cap M_n|}{|M_n|} = \prod_{f \in P} (1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}.$$

Now, suppose $n \in \mathbb{N}$ arbitrary. By division with remainder, we get $n + 1 = m t^{-d_P} + r$ with $0 \leq r < t^{-d_P}$. One defines $\hat{n} := n - r = m t^{-d_P} - 1$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|E_P \cap (M_n \setminus M_{\hat{n}})|}{|M_n|} &\leq \lim_{n \rightarrow \infty} \frac{|M_n| - |M_{\hat{n}}|}{|M_n|} = \\ &= \lim_{n \rightarrow \infty} \frac{(n + 1)^N - (n + 1 - r)^N}{(n + 1)^N} = 0, \end{aligned}$$

one has

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|E_P \cap M_n|}{|M_n|} &= \lim_{n \rightarrow \infty} \frac{|E_P \cap M_{\hat{n}}| + |E_P \cap (M_n \setminus M_{\hat{n}})|}{|M_n|} = \lim_{n \rightarrow \infty} \frac{|E_P \cap M_{\hat{n}}|}{|M_n|} \\ &= \lim_{n \rightarrow \infty} \frac{(n - r + 1)^N \prod_{f \in P} (1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}}{(n + 1)^N} = \\ &= \prod_{f \in P} (1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}. \end{aligned}$$

Easy computation leads to

$$\begin{aligned}
& (1 - t^{\deg(f)})^N + Nt^{\deg(f)}(1 - t^{\deg(f)})^{N-1} = \\
& = (1 - t^{\deg(f)})^{N-1}(1 + (N-1)t^{\deg(f)}) = \\
& = \left(1 - (N-1)t^{\deg(f)} + \binom{N-1}{2}t^{2\deg(f)} + \sum_{k=3}^N \tilde{\alpha}_k t^{k \cdot \deg(f)}\right) \cdot \\
& \cdot (1 + (N-1)t^{\deg(f)}) = \\
& = 1 + \frac{(N-1)(N-2) - 2(N-1)^2}{2} t^{2\deg(f)} + \sum_{k=3}^N \alpha_k t^{k \cdot \deg(f)} = \\
& = 1 - \binom{N}{2} t^{2\deg(f)} + \sum_{k=3}^N \alpha_k t^{k \cdot \deg(f)},
\end{aligned}$$

with coefficients $\tilde{\alpha}_k, \alpha_k \in \mathbb{N}$ that are independent of $\deg(f)$.

Define $H_f = \mathbb{F}[z]^N \setminus E_f$. Then

$$\lim_{n \rightarrow \infty} \frac{|H_f \cap M_n|}{|M_n|} = 1 - \lim_{n \rightarrow \infty} \frac{|E_f \cap M_n|}{|M_n|} = \binom{N}{2} t^{2\deg(f)} + \sum_{k=3}^N \alpha_k t^{k \cdot \deg(f)}.$$

Set $\alpha := \max(\alpha_3, \dots, \alpha_N)$ and let P_g be the set of all irreducible polynomials with degree at most g . Then $E_{P_g} \setminus E \subset \bigcup_{f \in \hat{P} \setminus P_g} H_f$ and consequently,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \frac{|(E_{P_g} \setminus E) \cap M_n|}{|M_n|} \leq \limsup_{n \rightarrow \infty} \frac{|(\bigcup_{f \in \hat{P} \setminus P_g} H_f) \cap M_n|}{|M_n|} \\
& \leq \limsup_{n \rightarrow \infty} \frac{\sum_{f \in \hat{P} \setminus P_g} |H_f \cap M_n|}{|M_n|} \\
& \leq \sum_{f \in \hat{P} \setminus P_g} \limsup_{n \rightarrow \infty} \frac{|H_f \cap M_n|}{|M_n|} = \sum_{f \in \hat{P} \setminus P_g} \left(\binom{N}{2} t^{2\deg(f)} + \sum_{k=3}^N \alpha_k t^{k \cdot \deg(f)} \right) = \\
& = \sum_{j=g+1}^{\infty} \varphi_j \left(\binom{N}{2} t^{2j} + \sum_{k=3}^N \alpha_k t^{k \cdot j} \right) \leq \sum_{j=g+1}^{\infty} t^{-j} \left(\binom{N}{2} t^{2j} + \sum_{k=3}^N \alpha_k t^{k \cdot j} \right) \\
& = \sum_{j=g+1}^{\infty} \binom{N}{2} t^j + \sum_{k=3}^N \alpha_k t^{(k-1) \cdot j} \leq \binom{N}{2} \frac{t^{g+1}}{1-t} + \alpha(N-2) \sum_{j=g+1}^{\infty} t^{2j} \xrightarrow{g \rightarrow \infty} 0.
\end{aligned}$$

Since $E \cap M_n = E_{P_g} \cap M_n \setminus ((E_{P_g} \setminus E) \cap M_n)$, one obtains

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{|E \cap M_n|}{|M_n|} &\geq \liminf_{n \rightarrow \infty} \frac{|E_{P_g} \cap M_n|}{|M_n|} - \limsup_{n \rightarrow \infty} \frac{|(E_{P_g} \setminus E) \cap M_n|}{|M_n|} \geq \\ &\geq \lim_{n \rightarrow \infty} \frac{|E_{P_g} \cap M_n|}{|M_n|} - \binom{N}{2} \frac{t^{g+1}}{1-t} + \alpha(N-2) \frac{t^{2(g+1)}}{1-t^2} \end{aligned}$$

as well as

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{|E \cap M_n|}{|M_n|} &\leq \limsup_{n \rightarrow \infty} \frac{|E_{P_g} \cap M_n|}{|M_n|} - \liminf_{n \rightarrow \infty} \frac{|(E_{P_g} \setminus E) \cap M_n|}{|M_n|} \\ &\leq \lim_{n \rightarrow \infty} \frac{|E_{P_g} \cap M_n|}{|M_n|}. \end{aligned}$$

It follows

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|E \cap M_n|}{|M_n|} &= \lim_{g \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{|E_{P_g} \cap M_n|}{|M_n|} = \\ &= \lim_{g \rightarrow \infty} \prod_{f \in P_g} ((1 - t^{\deg(f)})^N + N t^{\deg(f)} (1 - t^{\deg(f)})^{N-1}) = \\ &= \lim_{g \rightarrow \infty} \prod_{j=1}^g ((1 - t^j)^N + N t^j (1 - t^j)^{N-1})^{\varphi_j} = \\ &= \prod_{j=1}^{\infty} ((1 - t^j)^{N-1} (1 + (N-1)t^j))^{\varphi_j}. \end{aligned}$$

□

Corollary 3.4.

The natural density of N polynomials $d_1, \dots, d_N \in \mathbb{F}[z]$ to be pairwise coprime is equal to

$$1 - \binom{N}{2} t + \frac{1}{24} (N-1)(N-2)(3N^2 + 11N)t^2 + O(t^3).$$

Proof.

One has to show

$$\begin{aligned} \prod_{j=1}^{\infty} ((1 - t^j)^{N-1} (1 + (N-1)t^j))^{\varphi_j} &= \\ &= 1 - \binom{N}{2} t + \frac{1}{24} (N-1)(N-2)(3N^2 + 11N)t^2 + O(t^3). \end{aligned}$$

One uses the estimations $\varphi_j = \frac{1}{j}t^{-j} + O(t^{-(j-1)})$ as well as

$$\begin{aligned}
F_j &:= (1 - t^j)^{N-1}(1 + (N-1)t^j) = \\
&= 1 + \left(\binom{N-1}{2} - (N-1)^2 \right) t^{2j} + \\
&+ \left(\binom{N-1}{2}(N-1) - \binom{N-1}{3} \right) t^{3j} + O(t^{4j}) = \\
&= 1 + \frac{N-1}{2} \cdot (N-2 - 2(N-1))t^{2j} + \\
&+ (N-1)(N-2) \left(\frac{N-1}{2} - \frac{N-3}{6} \right) t^{3j} + O(t^{4j}) \\
&= 1 - \binom{N}{2} t^{2j} + \frac{1}{3} N(N-1)(N-2) t^{3j} + O(t^{4j}).
\end{aligned}$$

If one chooses x times the term with exponent $-kj$ (for $k \geq 2$) expanding $\prod_{j=1}^{\infty} F_j^{\varphi_j}$, one gets a term of the form $C(N) \binom{\varphi_j}{x} t^{xkj} = O(t^{(k-1)xj})$ with $C(N)$ only depending on N . Thus, one is only interested in the case that $k-1 = x = j = 1$ and in the case that one number from the set $\{k-1, x, j\}$ is equal to 2 and the others are equal to 1. In particular, the considered probability is equal to

$$\begin{aligned}
&\underbrace{\left(1 - \binom{N}{2} t^2 + \frac{1}{3} N(N-1)(N-2) t^3 \right)^{t^{-1}}}_{j=1, k \leq 3} \underbrace{\left(1 - \binom{N}{2} t^4 \right)^{\frac{1}{2}(t^{-2}-t^{-1})}}_{j=2, k \leq 2} + O(t^3) = \\
&= \left(1 - \underbrace{\binom{N}{2} t}_{k=2, x=1} + \underbrace{\frac{1}{3} N(N-1)(N-2) t^2}_{k=3, x=1} + \underbrace{\binom{t^{-1}}{2} \binom{N}{2}^2 t^4}_{k=2, x=2} + O(t^3) \right) \cdot \\
&\cdot \left(1 - \binom{N}{2} t^4 \cdot \frac{1}{2} t^{-2} + O(t^3) \right) + O(t^3) = \\
&= 1 - \binom{N}{2} t + \left(\frac{1}{3} N(N-1)(N-2) + \frac{N^2(N-1)^2}{8} - \frac{N(N-1)}{4} \right) t^2 + O(t^3) \\
&= 1 - \binom{N}{2} t + \left(\frac{1}{3} N(N-1)(N-2) + \frac{N(N-1)}{8} (N^2 - N - 2) \right) t^2 + O(t^3) \\
&= 1 - \binom{N}{2} t + \frac{1}{24} (N-1)(N-2)(3N^2 + 11N) t^2 + O(t^3),
\end{aligned}$$

which completes the proof of the corollary. \square

Corollary 3.4 leads to the same result as Theorem 3.2 with setting $N_1 = 0$, although different concepts of probability were used. This concordance could be explained in the following way: First, computing the natural density of pairwise coprimeness, those tuples of polynomials which contain a linear polynomial could be neglected. Moreover, the case that $d_i \equiv 0$ for some $i \in \{1, \dots, n\}$ could be neglected and hence, considering monic polynomials does not change the probability because two polynomials are coprime if and only if the corresponding monic polynomials are coprime. Thus, all degree dependencies of the considered coefficients in the asymptotic expansion could be neglected. Therefore, choosing the polynomials randomly with $\deg(d_i) \leq n_i$, the probability could be regarded as identical for all values $n_i \in \mathbb{N}$ since the set of polynomials with $\deg(d_i) \leq n_i$ is a disjoint union of the sets of polynomials whose degree is a fixed value less or equal to n_i . But the sets defined by the condition $\deg(d_i) \leq n_i$ form a subsequence of \mathcal{M}_n . Consequently, if one knows that the limit defining the natural density exists, one could conclude that it is equal to the constant value for this subsequence.

4 Mutual Left Coprimeness of Polynomial Matrices

The aim of this section is to compute the natural density of mutual left coprime polynomial matrices from $\mathbb{F}[z]^{m \times m}$ and compare it with the uniform probability that N nonsingular polynomial matrices are mutual left coprime, which was estimated in [6].

Theorem 4.1. [6, Theorem 12]

For $m, N \geq 2$ and $n_i \in \mathbb{N}$ for $i = 1, \dots, N$, the uniform probability that $D_i \in \mathbb{F}[z]^{m \times m}$ with $\deg(\det(D_i)) = n_i$ for $i = 1, \dots, N$ are mutually left coprime is

$$1 - \sum_{y=2}^{m+1} \binom{N}{y} t^m + O(t^{m+1}).$$

In the following, we want to compare the preceding result with the formula one gets for the natural density. It will turn out that, as in the previous section, the problem of computing the natural density could be reduced to

the calculation of the (uniform) probability that N constant matrices are mutually left coprime. Therefore, we start with the following definition:

Definition 4.2.

For $j \in \mathbb{N}$, the probability that $\mathcal{K}_N := \begin{bmatrix} K_1 & K_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & K_{N-1} & K_N \end{bmatrix}$ with $K_i \in (\mathbb{F}^j)^{m \times m}$ for $i = 1, \dots, N$ is of full row rank should be denoted by $W_j(N)$.

Theorem 4.3.

The natural density of N matrices $D_i \in \mathbb{F}[z]^{m \times m}$ for $i = 1, \dots, N$ to be mutually left coprime is equal to $\prod_{j=1}^{\infty} W_j(N)^{\varphi_j}$.

Proof.

Similar to the proof of Theorem 3.3, one obtains that the N matrices are mutually left coprime if and only if $\mathcal{D}_N/(f) := \mathcal{K}_N \in (\mathbb{F}^{\deg(f)})^{(N-1)m \times Nm}$ has full row rank for every (monic) irreducible polynomial $f \in \mathbb{F}[z]$. Denote the probability for this fact by W_f . As in the proof of Theorem 3.3 and with the notation from there, one gets $\lim_{n \rightarrow \infty} \frac{|E_P \cap M_n|}{|M_n|} = \prod_{f \in P} W_f$. According to that proof, one needs $W_f = 1 + O(t^{2 \deg(f)})$ to conclude $\lim_{n \rightarrow \infty} \frac{|E \cap M_n|}{|M_n|} = \prod_{j=1}^{\infty} W_j(N)^{\varphi_j}$. To this end, one shows that at least 2 of the matrices K_i have zero determinant if these matrices are not mutually left coprime. If $N = 2$, this clearly is true because the matrices are left coprime if not both of them have zero determinant. For $N \geq 3$, assume without restriction that $\det(K_N) \neq 0$ (otherwise permute the matrices K_i). Consequently, the columns of this matrix form a basis of $(\mathbb{F}^{\deg(f)})^m$ and adding appropriate linear combinations of the last m columns to the m preceding columns of

\mathcal{K}_N brings this matrix to the form $\begin{bmatrix} K_1 & K_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & K_{N-2} & K_{N-1} & 0 \\ 0 & 0 & 0 & 0 & K_N \end{bmatrix}$, which

is not left prime if and only if the submatrix consisting of the first $(N-2)m$ rows is not left prime. Per induction, it follows that at least two of the matrices K_1, \dots, K_{N-1} have zero determinant, which gives us the desired result. Furthermore, the probability that the determinant of K_i is equal to zero is $1 - t^{m^2 \deg(f)} |GL_m(\mathbb{F}^{\deg(f)})| = O(t^{\deg(f)})$. Thus, the proof is complete. \square

It remains to compute $W_j(N)$. To this end, we will firstly prove a recursion formula for it.

Lemma 4.4.

Let \hat{A} be the set of matrices K_i for which \mathcal{K}_N has full row rank and $\det(K_i) = 0$ for $i = 1, \dots, N$. Moreover, denote by $\hat{W}_j(N)$ the probability of \hat{A} . With $W_j(0) = W_j(1) = 1$, it holds for $N \geq 2$:

$$W_j(N) = \sum_{i=1}^N (-1)^{i-1} \binom{N}{i} \left(t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i W_j(N-i) + \hat{W}_j(N).$$

Proof.

If $\det(K_i) \neq 0$ for some $i \in \{1, \dots, N\}$, one sees as in the preceding proof that \mathcal{K}_N has full row rank if and only if the matrix $\mathcal{K}_{N-1}^{(i)}$ formed by the matrices from the set $\{K_1, \dots, K_N\} \setminus \{K_i\}$ has full row rank. Using the inclusion-exclusion-principle with \hat{A} and $A_i := \{\det(K_i) \neq 0 \text{ and } \mathcal{K}_{N-1}^{(i)} \text{ of full row rank}\}$, where $\hat{A} \cap A_i = \emptyset$, $P_i := \Pr(A_i)$ for $i = 1, \dots, N$ and $P_I := \Pr(\bigcap_{i \in I} A_i(N))$, one gets

$$W_j(N) = \sum_{I \subset \{1, \dots, N\}} (-1)^{|I|-1} P_I + \hat{W}_j(N).$$

With the same arguments as in the preceding proof, one obtains $\bigcap_{i \in I} A_i(N) = \{\det(K_i) \neq 0 \text{ for } i \in I \text{ and } \mathcal{K}_{N-|I|}^{(I)} \text{ has full row rank}\}$ and therefore, $P_I = \left(t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i W_j(N-i)$ for every I with $|I| = i$. Since there are $\binom{N}{i}$ subsets of cardinality i , the formula follows. \square

Corollary 4.5.

For $m \leq N-1$, it holds

$$W_j(N) = \sum_{i=1}^N (-1)^{i-1} \binom{N}{i} \left(t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i W_j(N-i).$$

Proof.

If $\det(K_i) = 0$ for $i = 1, \dots, N$, the column rank of \mathcal{K}_N is at most $Nm - N < Nm - m = (N-1)m$ and therefore, one has no full row rank. Consequently, $\hat{W}_j = 0$ and the statement follows from the preceding theorem. \square

To obtain a formula for $W_j(N)$ in the general case, one finally needs to calculate $\hat{W}_j(N)$.

Lemma 4.6.

For $j \in \mathbb{N}$ and $N \geq 2$, it holds:

$$\hat{W}_j(N) = \left(1 - t^{jm^2} |GL_m(\mathbb{F}^j)|\right)^N - \sum_{i=N-1}^{\min(m, N-1)} t^{j(m+1)} + O(t^{(m+2)j}).$$

Proof.

Denote by \tilde{W} the probability that $\det(K_i) = 0$ for $i = 1, \dots, N$ and \mathcal{K}_N is not of full row rank. We will show

$$\tilde{W} = \sum_{i=N-1}^{\min(m, N-1)} t^{j(m+1)} + O(t^{(m+2)j}).$$

The result follows since the sum of \tilde{W} and $\hat{W}_j(N)$ is equal to the probability that $\det(K_i) = 0$ for $i = 1, \dots, N$.

If $m < N - 1$, the probability that $\det(K_i) = 0$ for $i = 1, \dots, N$ is equal to $\left(1 - t^{jm^2} |GL_m(\mathbb{F}^j)|\right)^N = (1 - (1 - t^j + O(t^{j+1})))^N = O(t^{jN}) = O(t^{j(m+2)}),$

which is conform with $\sum_{i=N-1}^{\min(m, N-1)} t^{j(m+1)} = 0$ in this case.

Next, consider the case $m \geq N - 1$. We have to compute the probability that there exists $\xi \in (\mathbb{F}^j)^{1 \times m(N-1)} \setminus \{0\}$ with $\xi \mathcal{K}_N = 0$, i.e. that there exist $\xi_i \in (\mathbb{F}^j)^{1 \times m}$ for $i = 1, \dots, N - 1$ which are not all identically zero such that $\xi_1 K_1 = (\xi_1 + \xi_2) K_2 = \dots = (\xi_{N-2} + \xi_{N-1}) K_{N-1} = \xi_{N-1} K_N = 0$.

As in the proof for Lemma 10 of [6], one could show that either $\xi_i \neq 0$ for $i = 1, \dots, N-1$ and $\xi_i + \xi_{i+1} \neq 0$ for $i = 1, \dots, N-2$ or there exists $i \in \{1, \dots, N\}$ such that $\mathcal{K}_{N-1}^{(i)}$ formed by the matrices from the set $\{K_1, \dots, K_N\} \setminus \{K_i\}$ is not of full row rank. Per induction with respect to N , one knows that the probability for this is $O(t^{j(m+1)})$. Multiplication with the probability that $\det(K_i) = 0$, which is $O(t^j)$, leads to a term for the probability that is $O(t^{j(m+2)})$. Note that one could use induction since for $N = 2$, \tilde{W} is just equal to $1 - t^{j \cdot 2m^2} N(2m, m, m) = 1 - \prod_{l=m+1}^{2m} (1 - t^{jl})$ (see Lemma 2.9) because that $[K_1 \ K_2]$ is not of full row rank already implies $\det(K_1) = \det(K_2) = 0$. Thus, one could assume $\xi_i \neq 0$ and $\xi_i + \xi_{i+1} \neq 0$.

According to Lemma 2.9, the probability that $\dim(\ker(K_i)) = r_i$ is equal to

$$\begin{aligned}
t^{jm^2} \cdot N(m, m, m - r_i) &= t^{jmr_i} \cdot \prod_{l=r_i+1}^m (1 - t^{jl}) \prod_{l=0}^{m-(r_i+1)} \frac{t^{j(l-m)} - 1}{t^{-j(l+1)} - 1} = \\
&= t^{jmr_i} (1 + O(t^j)) \cdot \frac{\prod_{l=r_i+1}^m (t^{-jl} - 1)}{\prod_{l=1}^{m-r_i} (t^{-jl} - 1)} = \\
&= t^{jmr_i} (1 + O(t^j)) \cdot t^{-\frac{j}{2}(m(m+1) - r_i(r_i+1) - (m-r_i)(m-r_i+1))} \\
&= t^{jr_i^2} (1 + O(t^j)).
\end{aligned}$$

Fix $1 \leq r_i \leq m$ for $i = 1, \dots, N$. Then, the probability that $\dim(\ker(K_1)) = r_1$ is $t^{jr_1^2} \cdot (1 + O(t^j))$. For each such matrix K_1 , there are t^{-jr_1} possibilities for $\xi_1 \in (\mathbb{F}^j)^{1 \times m}$ with $\xi_1 K_1 = 0$. Furthermore, the probability that $\dim(\ker(K_2)) = r_2$ is $t^{jr_2^2} \cdot (1 + O(t^j))$ and for fixed ξ_1 and K_2 , there are t^{-jr_2} possibilities for $\xi_2 \in (\mathbb{F}^j)^{1 \times m}$ such that $(\xi_1 + \xi_2)K_2 = 0$. This procedure is continued until K_i and ξ_i are fixed for $i = 1, \dots, N-1$. As we assumed $\xi_{N-1} \neq 0$, the probability that $\xi_{N-1}K_N = 0$ is equal to t^{jm} .

Finally, one has to consider, which values for ξ_1, \dots, ξ_{N-1} lead to the same solutions for K_1, \dots, K_N . One clearly gets the same solutions if one multiplies ξ_i for $i = 1, \dots, N-1$ by the same scalar value, which effects a factor that is $O(t^j)$ for the probability. In summary, the overall probability is $O\left(t^{j(\sum_{i=1}^{N-1}(r_i^2 - r_i) + m+1)}\right) (1 + O(t^j))$. Hence, all cases in which $r_i \geq 2$ for some $i \in \{1, \dots, N-1\}$ could be neglected.

It remains to show that for $r_1 = \dots = r_N = 1$, only ξ_1, \dots, ξ_{N-1} which differ all by the same scalar factor lead to the same solutions for K_1, \dots, K_N . Then, one knows that the factor for the probability caused by this effect is exactly t^j and one gets a overall probability of $t^{(m+1)j} + O(t^{(m+2)j})$, which is conform with $\sum_{i=N-1}^{\min(m, N-1)} t^{j(m+1)} = t^{j(m+1)}$ in the considered case $m \geq N-1$.

To do this, we firstly show that the case that ξ_1, \dots, ξ_{N-1} are linearly dependent could be neglected. For the choice of such vectors ξ_i with the property that $\text{rk}[\xi_1^\top \dots \xi_{N-1}^\top] < N-1$ one has

$$O\left(\sum_{r=1}^{N-2} N(m, N-1, r)\right) = O\left(\sum_{r=1}^{N-2} t^{-jr(m+N-1-r)}\right) = O(t^{-j(N-2)(m+1)})$$

possibilities and for each of these possibilities the probability that $\xi_1 K_1 = (\xi_1 + \xi_2)K_2 = \dots = (\xi_{N-2} + \xi_{N-1})K_{N-1} = \xi_{N-1}K_N = 0$ is equal to t^{jNm} as $\xi_i \neq 0$ and $\xi_i + \xi_{i+1} \neq 0$. Additionally, one has again a factor of $O(t^j)$ because of the values for the vectors ξ_i that lead to the same solutions for K_1, \dots, K_N . In

summary, one gets a probability that is $O(t^{j(Nm+1-(N-2)(m+1))}) = O(t^{j(m+2)})$ since $-N \geq -m - 1$.

Hence, in the following, one could assume that ξ_1, \dots, ξ_{N-1} are linearly independent. If $\xi_1 K_1 = \tilde{\xi}_1 K_1 = 0$, $(\xi_1 + \xi_2)K_2 = (\tilde{\xi}_1 + \tilde{\xi}_2)K_2 = 0, \dots, \xi_{N-1} K_N = \tilde{\xi}_{N-1} K_N = 0$, it results from $r_1 = \dots = r_N = 1$ that there exist $\lambda_i \in \mathbb{F}^j$ with $\tilde{\xi}_1 = \lambda_1 \xi_1$, $\tilde{\xi}_i + \tilde{\xi}_{i+1} = \lambda_{i+1}(\xi_i + \xi_{i+1})$ for $i = 1, \dots, N-2$ and $\tilde{\xi}_{N-1} = \lambda_N \xi_{N-1}$. Since $\tilde{\xi}_1 - (\tilde{\xi}_1 + \tilde{\xi}_2) + \dots \pm (\tilde{\xi}_{N-2} + \tilde{\xi}_{N-1}) \mp \tilde{\xi}_{N-1} = 0$, it follows $(\lambda_1 - \lambda_2)\xi_1 + (\lambda_3 - \lambda_2)\xi_2 + \dots \pm (\lambda_{N-1} - \lambda_N)\xi_{N-1} = 0$. As ξ_1, \dots, ξ_{N-1} are linearly independent, this implies $\lambda_1 = \dots = \lambda_N$, which completes the proof of the whole theorem. \square

Now, we are able to solve the recursion formula of Lemma 4.4 to achieve an explicit expression for $W_j(N)$.

Theorem 4.7.

For $j \in \mathbb{N}$ and $N \geq 2$, the probability that N scalar matrices from $(\mathbb{F}^j)^{m \times m}$ are mutually left coprime is equal to

$$W_j(N) = 1 - \sum_{y=2}^{m+1} \binom{N}{y} t^{j(m+1)} + O(t^{j(m+2)}).$$

Proof.

This is shown per induction with respect to N . For $N = 2$, one just has to compute the probability that a rectangular matrix is of full rank. According to Lemma 2.9 with $n = 2m$ and $k = r = m$, this probability is equal to $\prod_{i=m+1}^{2m} (1 - (t^j)^i) = 1 - t^{j(m+1)} + O(t^{j(m+2)})$. Inserting the assumption of the induction into the first part of the recursion formula from Lemma 4.4, yields

$$\begin{aligned} & \sum_{i=1}^N (-1)^{i-1} \binom{N}{i} \left(t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i W_j(N-i) = \\ &= \sum_{i=1}^N (-1)^{i-1} \binom{N}{i} \left(t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i \left(1 - \sum_{y=2}^{m+1} \binom{N-i}{y} t^{j(m+1)} + O(t^{j(m+2)}) \right) \\ &= \sum_{i=1}^N (-1)^i \binom{N}{i} \left((-1) t^{jm^2} |GL_m(\mathbb{F}^j)| \right)^i + \\ &+ \sum_{i=1}^N (-1)^i \binom{N}{i} \sum_{y=2}^{m+1} \binom{N-i}{y} t^{j(m+1)} + O(t^{j(m+2)}) = \end{aligned}$$

$$\begin{aligned}
&= - \left(1 - t^{jm^2} |GL_m(\mathbb{F}^j)|\right)^N + 1 + \\
&+ \sum_{i=1}^{N-2} (-1)^i \binom{N}{i} \sum_{y=2}^{m+1} \binom{N-i}{y} t^{j(m+1)} + O(t^{j(m+2)}) = \\
&= - \left(1 - t^{jm^2} |GL_m(\mathbb{F}^j)|\right)^N + 1 - \sum_{y=2}^{\min(m+1, N-1)} \binom{N}{y} t^{j(m+1)} + O(t^{j(m+2)})
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{i=1}^{N-2} (-1)^i \binom{N}{i} \sum_{y=2}^{m+1} \binom{N-i}{y} = \sum_{i=1}^{N-2} \sum_{y=2}^{\min(m+1, N-i)} (-1)^i \frac{N!}{i! \cdot y! \cdot (N-i-y)!} \\
&= \sum_{y=2}^{\min(m+1, N-1)} \sum_{i=1}^{N-y} (-1)^i \frac{N!}{i! \cdot y! \cdot (N-i-y)!} = - \sum_{y=2}^{\min(m+1, N-1)} \binom{N}{y},
\end{aligned}$$

where the last step follows from Lemma 2.13. Using the formula for $\hat{W}_j(N)$ from the preceding lemma, one obtains

$$\begin{aligned}
W_j(N) &= - \left(1 - t^{jm^2} |GL_m(\mathbb{F}^j)|\right)^N + 1 - \sum_{y=2}^{\min(m+1, N-1)} \binom{N}{y} t^{j(m+1)} + \hat{W}_j(N) \\
&= 1 - \left(\sum_{i=N-1}^{\min(m, N-1)} 1 + \sum_{y=2}^{\min(m+1, N-1)} \binom{N}{y} \right) t^{j(m+1)} + O(t^{j(m+2)}) = \\
&= 1 - \sum_{y=2}^{m+1} \binom{N}{y} t^{j(m+1)} + O(t^{j(m+2)}).
\end{aligned}$$

The last equality is valid because if $m+1 \leq N-1$, it holds $\sum_{i=N-1}^{\min(m, N-1)} 1 = 0$ and if $m+1 > N-1$, it holds $\sum_{i=N-1}^{\min(m, N-1)} 1 + \sum_{y=2}^{\min(m+1, N-1)} \binom{N}{y} = 1 + \sum_{y=2}^{N-1} \binom{N}{y} = \sum_{y=2}^N \binom{N}{y} = \sum_{y=2}^{m+1} \binom{N}{y}$. \square

Theorem 4.8.

The natural density of $D_i \in \mathbb{F}[z]^{m \times m}$ for $i = 1, \dots, N$ to be mutually left coprime is equal to

$$\prod_{j=1}^{\infty} \left(1 - \sum_{y=2}^{m+1} \binom{N}{y} t^{j(m+1)} + O(t^{j(m+2)}) \right)^{\varphi_j} = 1 - \sum_{y=2}^{m+1} \binom{N}{y} t^m + O(t^{m+1}).$$

5 Conclusion

We computed the natural density of mutually left coprime polynomial matrices and compared the result with the uniform probability of mutual left coprimeness. If the considered matrices are scalar, i.e. for the case of pairwise coprime polynomials, we could even show a more precise estimation than in the general case. It is remarkable that probability and natural density asymptotically coincide in all considered cases. However, the exact values for these two concepts of probability might differ. For the case of pairwise coprimeness of scalar polynomials, we have already seen that the coefficient of t^2 depends on the degrees of the constituent polynomials and is different from the coefficient of t^2 in the series expansion of the formula for the natural density if $N_1 \neq 0$. Moreover, it is not difficult to see that further coefficients will also depend on the degrees of the involved polynomials.

For $m \geq 2$, the exact value for the uniform probability depends on the degrees of the determinants of the constituent matrices and therefore, does not coincide with the natural density for each degree structure. Consider for example the case $m = 2$ and $\deg(\det(D_i)) = 1$ for $i = 1, 2$. Easy computation yields that the uniform probability of left coprimeness is equal to $1 - \frac{1}{t^{-2}+t^{-1}} = 1 - t^2 \sum_{k=0}^{\infty} (-t)^k$, which is larger than the natural density being equal to $(1 - t^2)(1 - t^3)$. One could expect that with increasing the values n_i , the number of coinciding coefficients between uniform probability and natural density increases. But it is still an open question if the uniform probability of mutual left coprimeness tends to the value of the natural density if $n_i \rightarrow \infty$ for $i = 1, \dots, N$.

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